Name S. Jidarian

Day / Period ____________________

Union County Vocational-Technical Schools: Magnet High School

AP Calculus-I/AB Final Project
(Reading Ahead for Calculus-II/BC)

June 1, 2015

Use of an engineering calculator is recommended.
Please show steps as you solve each problem.

1-Complete the table using the exact particular solution of the differential equation and its approximation using the Euler's Method.

$$\frac{dy}{dx} = \frac{x}{2y}; \quad y(0) = 2$$

n	0	1	2	3
x	0	0.2	0.4	0.6
y = f(x) Exact	2	2.0050	2.0199	2.0445
y = f(x), h=0.2	2	2.0000	2.0100	2.0299

Euler's Method

$x_0 = 0$, $y_0 = 2$ $\Longrightarrow$ $m_0 = \left.\frac{dy}{dx}\right|_{(0,2)} = \frac{0}{2(2)} = 0$, $m = \frac{x}{2y}$

$n=1 \Rightarrow$ $y_1 = y_0 + m_0 h = 2 + 0(0.2) = 2.0000 = y_1$

$\therefore$ $x_1 = 0.2$, $y_1 = 2.0000$

$n = 2 \Rightarrow$ $x_2 = 0.4$, $m_1 = \left.\frac{dy}{dx}\right|_{(0.2,\, 2.000)} = \frac{0.2}{2(2.0000)} = 0.05 = m_1$

$y_2 = y_1 + m_1 h = 2.0000 + 0.05(0.2) = 2.0100 = y_2$

$n = 3 \Rightarrow$ $x_3 = 0.6$, $m_2 = \left.\frac{dy}{dx}\right|_{(0.4,\, 2.0100)} = \frac{0.4}{2(2.0100)} = 0.0995 = m_2$

$y_3 = y_2 + m_2 h = 2.0100 + 0.0995(0.2) = 2.0299 = y_3$

Exact Solution

$$\frac{dy}{dx} = \frac{x}{2y} \implies dy = y'\,dx \rightarrow dy = \frac{x}{2y}\,dx$$

or

$$2y\,dy = x\,dx \leftarrow \text{separation of variables}$$

$$\int 2y\,dy = \int x\,dx \implies y^2 = \frac{1}{2}x^2 + c$$

$$y = \pm\sqrt{\frac{1}{2}x^2 + c}$$

given $y(0) = 2$, we'll choose the "+" branch

$$y = \sqrt{\frac{1}{2}x^2 + c}$$

$$y(0) = 2 \implies 2 = \sqrt{\frac{1}{2}(0)^2 + c} \rightarrow 2 = \sqrt{c} \implies \boxed{c = 4}$$

$$\boxed{y = \sqrt{\frac{1}{2}x^2 + 4}} \leftarrow \text{Particular Solution when } y(0) = 2$$

Now fill in the table for given x-values:

$y(0) = 2$
$y(0.2) = 2.0050$
$y(0.4) = 2.0199$
$y(0.6) = 2.0445$

Exact

2-According to United Nations data, the world population at the beginning of 1990 was 5.2 billion and growing at a rate of 1.95% per year. Assuming an exponential growth model:

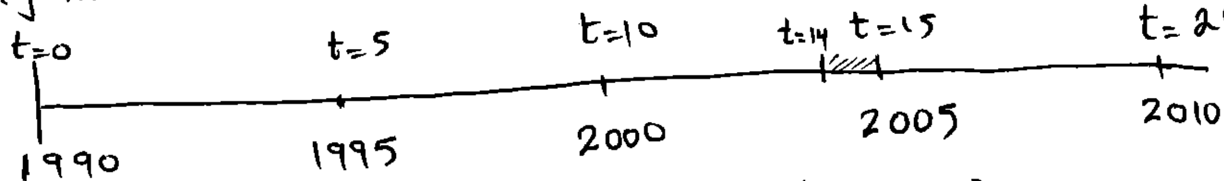
* (a) Estimate the world population at the beginning of year 2005, and

(b) Find the year in which the population will double.

(Note: the information has been modified, and is not representative of the original UN data.)

(a) $\frac{dP}{dt} = 0.0195P \rightarrow P = P_o e^{0.0195t}$

Using the initial condition $(t=0, P=5.4) \Rightarrow \boxed{P(t) = 5.4e^{0.0195t}}$



Beginning of year 15 : $t=14$ (end of year 4)

$P(14) = 5.4e^{0.0195(14)} = 7.09506 \approx 7.1$ billion

(b) $P(t) = 2P_o \Rightarrow 2P_o = P_o e^{0.0195t} \Rightarrow e^{0.0195t} = 2$

$0.0195t = \ln(2) \rightarrow t = 35.546$ yrs

$\therefore$ Population double during the year $1990+35 = \boxed{2025}$

---

* Please Note:

strictly speaking, the 1.95% is the "discrete" or the ~~annual~~ annualized growth rate. To find the equivalent "Continuous" growth rate, we need to do the following conversion:

$P_1 = P_o(1.0195)$

$\Rightarrow 1.0195 P_o = P_o e^{k(1)} \Rightarrow e^{k} = 1.0195 \Rightarrow k = 0.0193$

Therefore $\frac{dP}{dt} = 0.0193P \Rightarrow P = P_o e^{0.0193t}$

or $\boxed{P(t) = 5.4e^{0.0193t}}$

slightly different

3-A polar bear cub that weighs 20 pounds at birth gains weight at the rate of $\frac{dw}{dt} = k(850 - w)$ where **k** is a constant. The cub reaches 210 pounds in one year.

(a) solve the differential equation for the particular solution (weight as a function of time)

(b) how many years will it take for the animal to reach 700 pounds?

(c) determine the maximum weight the animal can ever achieve using two different logics; justify your answers.

(a) $\frac{dw}{850-w} = k\,dt \Rightarrow \int \frac{dw}{850-w} = \int k\,dt \Rightarrow \ln(850-w) = -kt + C_1$

Note that we don't need to use absolute value since $850-w>0$

$$850 - W = e^{-kt+C_1} = e^{C_1} \cdot e^{-kt} = Ce^{-kt} \quad ; \ C>0$$

$$850 - W = Ce^{-kt} \Rightarrow \boxed{W = 850 - Ce^{-kt}}$$

for the initial condition $(t=0, W=20)$:

$$20 = 850 - Ce^{-k(0)} = 850 - C \Rightarrow \boxed{C = 830}$$

$$\therefore \boxed{W(t) = 850 - 830e^{-kt}}$$

for condition: $t=1$, $W=210$ ← calculate k

$$210 = 850 - 830e^{-k(1)}$$

$$830e^{-k} = 850 - 210 \Rightarrow 830e^{-k} = 640$$

$$e^{-k} = \frac{640}{830} \Rightarrow -k = \ln\left(\frac{640}{830}\right)$$

$$k = 0.259958 \quad \text{or} \quad k = 0.2600$$

$$\therefore \boxed{W(t) = 850 - 830e^{-0.26t}}$$

(b) $W(t) = 700$ Pounds ; $t = ?$

$700 = 850 - 830\, e^{-0.2600 t}$

$\boxed{t = 6.58 \text{ years}}$

(c) Method-1

$\lim_{t \to \infty} W(t) = 850$

Method-2

$\frac{dW}{dt} = k(850 - W)$

$\frac{dW}{dt}$ = growth rate $= 0$ @ $W = 850$

(Stops growing)

4-A Logistic Differential Equation is defined as: $\frac{dy}{dt} = ky(1 - \frac{y}{L})$, where k and L are positive constants. Its general solution is given by: $y = \frac{L}{1 + b.e^{-kt}}$.

L is called the carrying capacity, and k is a proportionality constant.

(a) Find the particular solution to $\frac{dy}{dt} = \frac{7y}{80} - \frac{y^2}{2400}$ with initial condition (t=0 , y=16). The time variable **t** is measured in years.

(b) Determine the population size y after six (6) years.

(c) Determine the population size and the corresponding time when growth rate of the population y is maximum.

(d) Use Euler's method to approximate y(1) in one step. Would this be an over- or under-estimation of the actual value? Explain why.

(a) $\frac{dy}{dt} = \frac{7y}{80} - \frac{y^2}{2400} = \frac{7}{80} y \left(1 - \frac{y}{210}\right) \leftarrow$ standard form

$\therefore \quad k = \frac{7}{80} = 0.0875$

$L = 210$

$y = \frac{210}{1 + b\, e^{-0.0875t}}$

for $(0,16) \Rightarrow \quad 16 = \frac{210}{1+b} \Rightarrow 1+b = \frac{210}{16} \Rightarrow \boxed{b = 12.125}$

or $\boxed{y = \frac{210}{1 + 12.125\, e^{-0.0875t}}}$

(b) $\boxed{y(6) = 25.69 \approx 26}$

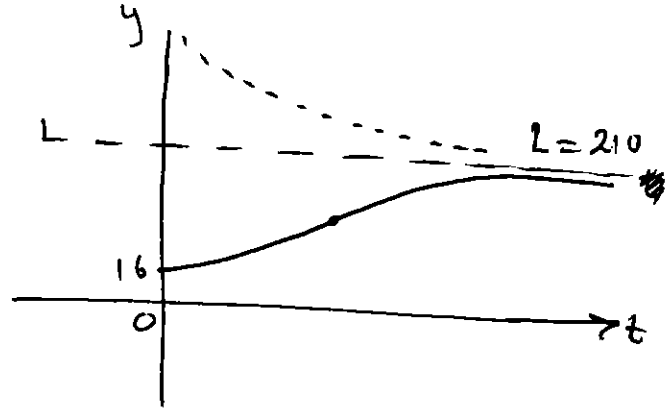
(c) Let $g(y) = \frac{dy}{dt} = \frac{7}{80} y \left(1 - \frac{y}{210}\right) \leftarrow$ growth rate as a function of population (y)

$\frac{dg}{dy} = \frac{7}{80}\left[\left(1 - \frac{y}{210}\right) - \frac{y}{210}\right] = \frac{7}{80}\left(1 - \frac{2y}{210}\right) = 0$ Set $= 0$

$\Rightarrow 1 - \frac{2y}{210} = 0 \rightarrow \boxed{y = \frac{210}{2} = 105}$

(c) continued

It is easily verified that **maximum** growth rate happens at $y = 105$ (See the diagram, graph of the population)



In general, in a growing Logistic function, maximum growth rate happens at $y = \frac{L}{2}$

(in this case at $y = \frac{210}{2} = 105$)

(d) $t_0 = 0$ $\quad h = 1$ (step size)

$y_0 = 16$

$$m_0 = \left.\frac{dy}{dt}\right|_{t=0} = \left.\frac{dy}{dt}\right|_{y=16} = \frac{7}{80}(16)\left(1 - \frac{16}{210}\right) = 1.2933$$

$$y(1) = y_1 = y_0 + m_0 h = 16 + 1.2933(1) = 17.2933$$

$$\boxed{y(1) \approx 17.2933}$$

Note that the exact value of $y(1) = \cancel{16.1028}$ 17.3423

Obviously the Euler's approximation under-estimates $y(1)$. This was expected since @ $t = 1$, $y(t)$ is concave up in which tangent line approximation falls below the actual curve.

5-Find the particular solution of the differential equation $y' + \frac{2x}{x^2-9}y = x$ that satisfies the boundary condition $y(0) = -1$

1st Order Linear Diff Eqn: $y' + P(x)y = Q(x)$

$P(x) = \frac{2x}{x^2-9}$, $Q(x) = x$

$u(x) = e^{\int P(x)\,dx} = e^{\int \frac{2x}{x^2-9}dx} = e^{\ln|x^2-9|}$

$= |x^2-9| \longrightarrow$ use x^2-9

$u(x) = x^2-9$

$y = \frac{1}{u(x)}\int Q(x)\,u(x)\,dx$

$= \frac{1}{x^2-9}\int x(x^2-9)\,dx = \frac{1}{2(x^2-9)}\int (2x)(x^2-9)\,dx$

$y = \frac{1}{2(x^2-9)}\left[\frac{(x^2-9)^2}{2} + c\right]$

for $y(0) = -1$

$-1 = \frac{1}{2(0-9)}\left[\frac{(0-9)^2}{2} + c\right] \Rightarrow -1 = \frac{1}{-18}\left[\frac{81}{2} + c\right]$

$\frac{81}{2} + c = 18 \rightarrow c = 18 - \frac{81}{2} = -\frac{45}{2}$

$$\boxed{y = \frac{1}{2(x^2-9)}\left[\frac{(x^2-9)^2}{2} - \frac{45}{2}\right]}$$

6- Find the orthogonal trajectories for the family of curves represented by $2x^2 + 4y^2 = C$, where C is a constant.

(1) what is $\frac{dy}{dx}$ for the original family

$$4x + 8y\frac{dy}{dx} = 0 \longrightarrow \frac{dy}{dx} = -\frac{4x}{8y}$$

or $$\frac{dy}{dx} = \frac{-x}{2y}$$

(2) what is $\frac{dy}{dx}$ for the orthogonal trajectories

$$\frac{dy}{dx} = \frac{2y}{x}$$ (negative reciprocal)

(3) Solve the D.E. for the orthogonal trajectories

$$dy = \frac{2y}{x}\,dx$$

$$\frac{dy}{y} = \frac{2\,dx}{x} \Longrightarrow \int\frac{dy}{y} = 2\int\frac{dx}{x}$$

$$\ln|y| = 2\ln|x| + C_1 \Rightarrow \ln|y| = \ln|x|^2 + \ln C_2 \;;\; C_2 > 0$$

or $$\ln|y| = \ln(C_2|x|^2) = \ln(C_2x^2) \Rightarrow |y| = Cx^2$$

$$y = \pm C_2x^2$$ or

$$\boxed{y = Cx^2}$$