Euler's Method

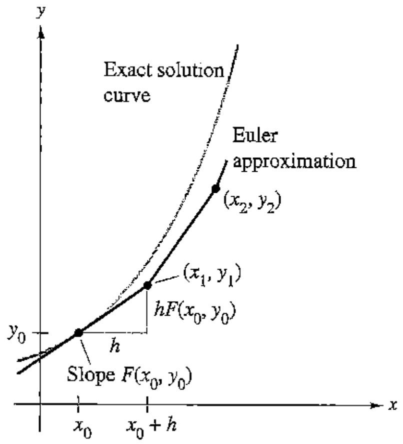


Figure 6.6

Euler's Method is a numerical approach to approximating the particular solution of the differential equation

$$y' = F(x, y)$$

that passes through the point (x_0, y_0). From the given information, you know that the graph of the solution passes through the point (x_0, y_0) and has a slope of $F(x_0, y_0)$ at this point. This gives you a "starting point" for approximating the solution.

From this starting point, you can proceed in the direction indicated by the slope. Using a small step h, move along the tangent line until you arrive at the point (x_1, y_1), where

$$x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + hF(x_0, y_0)$$

as shown in Figure 6.6. If you think of (x_1, y_1) as a new starting point, you can repeat the process to obtain a second point (x_2, y_2). The values of x_i and y_i are as follows.

$$\begin{aligned}
x_1 &= x_0 + h & y_1 &= y_0 + hF(x_0, y_0)\\
x_2 &= x_1 + h & y_2 &= y_1 + hF(x_1, y_1)\\
&\vdots & &\vdots\\
x_n &= x_{n-1} + h & y_n &= y_{n-1} + hF(x_{n-1}, y_{n-1})
\end{aligned}$$

NOTE You can obtain better approximations of the exact solution by choosing smaller and smaller step sizes.

EXAMPLE 6 Approximating a Solution Using Euler's Method

Use Euler's Method to approximate the particular solution of the differential equation

$$y' = x - y$$

passing through the point $(0, 1)$. Use a step of $h = 0.1$.

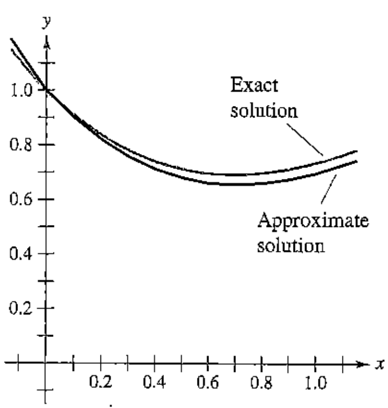


Figure 6.7

Solution Using $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x - y$, you have $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3, \ldots,$ and

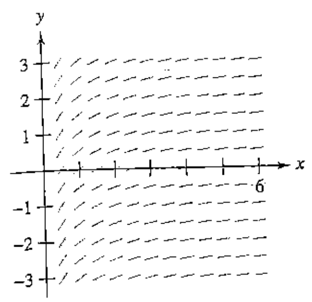
$$\begin{aligned}
y_1 &= y_0 + hF(x_0, y_0) = 1 + (0.1)(0 - 1) = 0.9\\
y_2 &= y_1 + hF(x_1, y_1) = 0.9 + (0.1)(0.1 - 0.9) = 0.82\\
y_3 &= y_2 + hF(x_2, y_2) = 0.82 + (0.1)(0.2 - 0.82) = 0.758.
\end{aligned}$$

The first ten approximations are shown in the table. You can plot these values to see a graph of the approximate solution, as shown in Figure 6.7.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	1	0.900	0.820	0.758	0.712	0.681	0.663	0.657	0.661	0.675	0.697

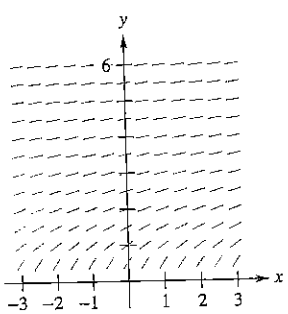
NOTE For the differential equation in Example 6, you can verify the exact solution to be $y = x - 1 + 2e^{-x}$. Figure 6.7 compares this exact solution with the approximate solution obtained in Example 6.

61. ***Slope Field*** Use the slope field for the differential equation $y' = 1/x$, where $x > 0$, to sketch the graph of the solution that satisfies each given initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/x$ as $x \to \infty$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) $(1, 0)$ (b) $(2, -1)$

62. ***Slope Field*** Use the slope field for the differential equation $y' = 1/y$, where $y > 0$, to sketch the graph of the solution that satisfies each initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/y$ as $x \to \infty$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) $(0, 1)$ (b) $(1, 1)$

Slope Fields **In Exercises 63–68, use a computer algebra system to (a) graph the slope field for the differential equation and (b) graph the solution satisfying the specified initial condition.**

63. $\dfrac{dy}{dx} = 0.5y, \quad y(0) = 6$

64. $\dfrac{dy}{dx} = 2 - y, \quad y(0) = 4$

65. $\dfrac{dy}{dx} = 0.02y(10 - y), \quad y(0) = 2$

66. $\dfrac{dy}{dx} = 0.2x(2 - y), \quad y(0) = 9$

67. $\dfrac{dy}{dx} = 0.4y(3 - x), \quad y(0) = 1$

68. $\dfrac{dy}{dx} = \dfrac{1}{2}e^{-x/8} \sin \dfrac{\pi y}{4}, \quad y(0) = 2$

Euler's Method **In Exercises 69–74, use Euler's Method to make a table of values for the approximate solution of the differential equation with the specified initial value. Use n steps of size h.**

69. $y' = x + y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1$

70. $y' = x + y, \quad y(0) = 2, \quad n = 20, \quad h = 0.05$

71. $y' = 3x - 2y, \quad y(0) = 3, \quad n = 10, \quad h = 0.05$

72. $y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4$

73. $y' = e^{xy}, \quad y(0) = 1, \quad n = 10, \quad h = 0.1$

74. $y' = \cos x + \sin y, \quad y(0) = 5, \quad n = 10, \quad h = 0.1$

In Exercises 75–77, complete the table using the exact solution of the differential equation and two approximations obtained using Euler's Method to approximate the particular solution of the differential equation. Use $h = 0.2$ and 0.1 and compute each approximation to four decimal places.

x	0	0.2	0.4	0.6	0.8	1.0
$y(x)$ (exact)						
$y(x)$ ($h = 0.2$)						
$y(x)$ ($h = 0.1$)						

	Differential Equation	*Initial Condition*	*Exact Solution*
75.	$\dfrac{dy}{dx} = y$	$(0, 3)$	$y = 3e^x$
76.	$\dfrac{dy}{dx} = \dfrac{2x}{y}$	$(0, 2)$	$y = \sqrt{2x^2 + 4}$
77.	$\dfrac{dy}{dx} = y + \cos(x)$	$(0, 0)$	$y = \frac{1}{2}(\sin x - \cos x + e^x)$

78. Compare the values of the approximations in Exercises 75–77 with the values given by the exact solution. How does the error change as h increases?

79. ***Temperature*** At time $t = 0$ minutes, the temperature of an object is 140°F. The temperature of the object is changing at the rate given by the differential equation

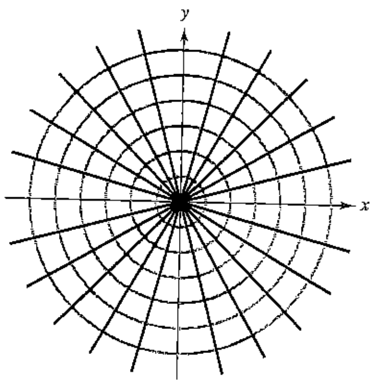
$$\frac{dy}{dt} = -\frac{1}{2}(y - 72).$$

(a) Use a graphing utility and Euler's Method to approximate the particular solutions of this differential equation at $t = 1$, 2, and 3. Use a step size of $h = 0.1$. (A graphing utility program for Euler's Method is available on the website *college.hmco.com*.)

(b) Compare your results with the exact solution

$$y = 72 + 68e^{-t/2}.$$

80. ***Temperature*** Repeat Exercise 79 using a step size of $h = 0.05$. Compare the results.



Each line $y = Kx$ is an orthogonal trajectory to the family of circles.
Figure 6.15

A common problem in electrostatics, thermodynamics, and hydrodyn involves finding a family of curves, each of which is orthogonal to all membe given family of curves. For example, Figure 6.15 shows a family of circles

$$x^2 + y^2 = C \qquad \text{Family of circles}$$

each of which intersects the lines in the family

$$y = Kx \qquad \text{Family of lines}$$

at right angles. Two such families of curves are said to be **mutually orthogonal** each curve in one of the families is called an **orthogonal trajectory** of the family. In electrostatics, lines of force are orthogonal to the *equipotential cu* In thermodynamics, the flow of heat across a plane surface is orthogonal t *isothermal curves*. In hydrodynamics, the flow (stream) lines are orthog trajectories of the *velocity potential curves*.

EXAMPLE 8 Finding Orthogonal Trajectories

Describe the orthogonal trajectories for the family of curves given by

$$y = \frac{C}{x}$$

for $C \neq 0$. Sketch several members of each family.

Solution First, solve the given equation for C and write $xy = C$. Then, by differ tiating implicitly with respect to x, you obtain the differential equation

$$xy' + y = 0 \qquad \text{Differential equation}$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}. \qquad \text{Slope of given family}$$

Because y' represents the slope of the given family of curves at (x, y), it follows t the orthogonal family has the negative reciprocal slope x/y. So,

$$\frac{dy}{dx} = \frac{x}{y}. \qquad \text{Slope of orthogonal family}$$

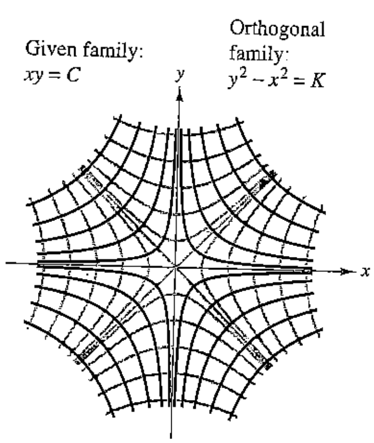
Now you can find the orthogonal family by separating variables and integrating.

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = K$$

The centers are at the origin, and the transverse axes are vertical for $K > 0$ an horizontal for $K < 0$. If $k = 0$, the orthogonal trajectories are the lines $y = \pm x$. I $K \neq 0$, the orthogonal trajectories are hyperbolas. Several trajectories are shown in Figure 6.16.

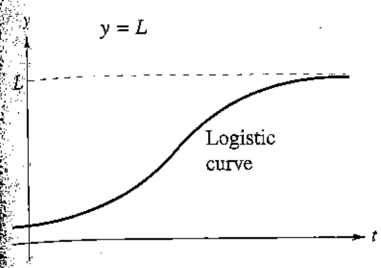


Orthogonal trajectories
Figure 6.16

Logistic Differential Equation

In Section 6.2, the exponential growth model is derived from the fact that the rate of change of a variable y is proportional to the value of y. You observed that the differential equation $dy/dt = ky$ has the general solution $y = Ce^{kt}$. Exponential growth is unlimited, but when describing a population, there often exists some upper limit L past which growth cannot occur. This upper limit L is called the **carrying capacity,** which is the maximum population $y(t)$ that can be sustained or supported as time t increases. A model that is often used for this type of growth is the **logistic differential equation**

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Logistic differential equation}$$



Note that as $t \to \infty$, $y \to L$.
Figure 6.17

where k and L are positive constants. A population that satisfies this equation does not grow without bound, but approaches the carrying capacity L as t increases.

From the equation, you can see that if y is between 0 and the carrying capacity L, then $dy/dt > 0$, and the population increases. If y is greater than L, then $dy/dt < 0$, and the population decreases. The graph of the function y is called the *logistic curve*, as shown in Figure 6.17.

EXAMPLE 9 Deriving the General Solution

Solve the logistic differential equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right)$.

Solution Begin by separating variables.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Write differential equation.}$$

$$\frac{1}{y(1 - y/L)}dy = kdt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y(1 - y/L)}dy = \int kdt \qquad \text{Integrate each side.}$$

just accept it for now!

$$\int \left(\frac{1}{y} + \frac{1}{L - y}\right)dy = \int kdt \qquad \text{Rewrite left side using partial fractions.}$$

$$\ln|y| - \ln|L - y| = kt + C \qquad \text{Find antiderivative of each side.}$$

$$\ln\left|\frac{L - y}{y}\right| = -kt - C \qquad \text{Multiply each side by } -1 \text{ and simplify.}$$

$$\left|\frac{L - y}{y}\right| = e^{-kt-C} = e^{-C}e^{-kt} \qquad \text{Exponentiate each side.}$$

$$\frac{L - y}{y} = be^{-kt} \qquad \text{Let } \pm e^{-C} = b.$$

Solving this equation for y produces $y = \dfrac{L}{1 + be^{-kt}}$.

From Example 9, you can conclude that all solutions of the logistic differential equation are of the general form

$$y = \frac{L}{1 + be^{-kt}}.$$

will cover later in Calc-2

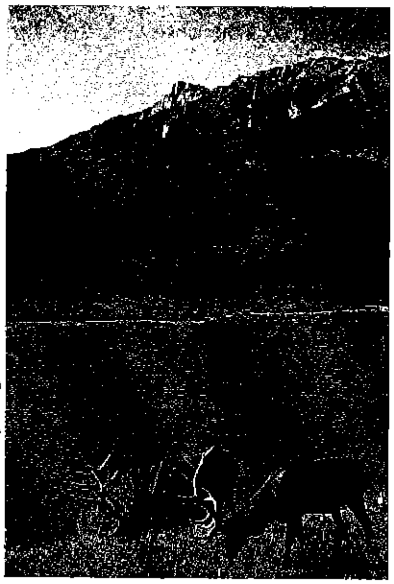
EXPLORATION

Use a graphing utility to investigate the effects of the values of L, b, and k on the graph of

$$y = \frac{L}{1 + be^{-kt}}.$$

Include some examples to support your results.

EXAMPLE 10 Solving a Logistic Differential Equation



John Edwards/Getty Images

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right), \quad 40 \le p \le 4000$$

where t is the number of years.

a. Write a model for the elk population in terms of t.

b. Graph the slope field of the differential equation and the solution that passes through the point $(0, 40)$.

c. Use the model to estimate the elk population after 15 years.

d. Find the limit of the model as $t \to \infty$.

EXPLORATION

Explain what happens if $p(0) = L$.

Solution

a. You know that $L = 4000$. So, the solution of the equation is of the form

$$p = \frac{4000}{1 + be^{-kt}}.$$

Because $p(0) = 40$, you can solve for b as shown.

$$40 = \frac{4000}{1 + be^{-k(0)}}$$

$$40 = \frac{4000}{1 + b} \quad \Longrightarrow \quad b = 99$$

Then, because $p = 104$ when $t = 5$, you can solve for k.

$$104 = \frac{4000}{1 + 99e^{-k(5)}} \quad \Longrightarrow \quad k \approx 0.194$$

So, a model for the elk population is given by $p = \dfrac{4000}{1 + 99e^{-0.194t}}$.

b. Using a graphing utility, you can graph the slope field of

$$\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$$

and the solution that passes through $(0, 40)$, as shown in Figure 6.18.

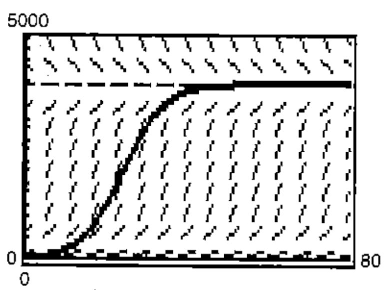


Figure 6.18

c. To estimate the elk population after 15 years, substitute 15 for t in the model

$$p = \frac{4000}{1 + 99e^{-0.194(15)}} \qquad \text{Substitute 15 for } t.$$

$$= \frac{4000}{1 + 99e^{-2.91}} \approx 626 \qquad \text{Simplify.}$$

d. As t increases without bound, the denominator of $\dfrac{4000}{1 + 99e^{-0.194t}}$ gets closer to 1.

So, $\displaystyle\lim_{t\to\infty} \frac{4000}{1 + 99e^{-0.194t}} = 4000.$

Exercises for Section 6.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, find the general solution of the differential equation.

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$
2. $\dfrac{dy}{dx} = \dfrac{x^2 + 2}{3y^2}$
3. $\dfrac{dr}{ds} = 0.05r$
4. $\dfrac{dr}{ds} = 0.05s$
5. $(2 + x)y' = 3y$
6. $xy' = y$
7. $yy' = \sin x$
8. $yy' = 6\cos(\pi x)$
9. $\sqrt{1 - 4x^2}\,y' = x$
10. $\sqrt{x^2 - 9}\,y' = 5x$
11. $y \ln x - xy' = 0$
12. $4yy' - 3e^x = 0$

In Exercises 13–22, find the particular solution that satisfies the initial condition.

Differential Equation	*Initial Condition*
13. $yy' - e^x = 0$	$y(0) = 4$
14. $\sqrt{x} + \sqrt{y}\,y' = 0$	$y(1) = 4$
15. $y(x + 1) + y' = 0$	$y(-2) = 1$
16. $2xy' - \ln x^2 = 0$	$y(1) = 2$
17. $y(1 + x^2)y' - x(1 + y^2) = 0$	$y(0) = \sqrt{3}$
18. $y\sqrt{1 - x^2}\,y' - x\sqrt{1 - y^2} = 0$	$y(0) = 1$
19. $\dfrac{du}{dv} = uv \sin v^2$	$u(0) = 1$
20. $\dfrac{dr}{ds} = e^{r-2s}$	$r(0) = 0$
21. $dP - kP\,dt = 0$	$P(0) = P_0$
22. $dT + k(T - 70)\,dt = 0$	$T(0) = 140$

In Exercises 23 and 24, find an equation of the graph that passes through the point and has the given slope.

23. $(1, 1),\quad y' = -\dfrac{9x}{16y}$

24. $(8, 2),\quad y' = \dfrac{2y}{3x}$

In Exercises 25 and 26, find all functions f having the indicated property.

25. The tangent to the graph of f at the point (x, y) intersects the x-axis at $(x + 2, 0)$.

26. All tangents to the graph of f pass through the origin.

In Exercises 27–34, determine whether the function is homogeneous, and if it is, determine its degree.

27. $f(x, y) = x^3 - 4xy^2 + y^3$
28. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$
29. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$
30. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$
31. $f(x, y) = 2 \ln xy$
32. $f(x, y) = \tan(x + y)$
33. $f(x, y) = 2 \ln \dfrac{x}{y}$
34. $f(x, y) = \tan \dfrac{y}{x}$

In Exercises 35–40, solve the homogeneous differential equation.

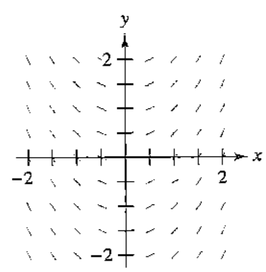
35. $y' = \dfrac{x + y}{2x}$
36. $y' = \dfrac{x^3 + y^3}{xy^2}$
37. $y' = \dfrac{x - y}{x + y}$
38. $y' = \dfrac{x^2 + y^2}{2xy}$
39. $y' = \dfrac{xy}{x^2 - y^2}$
40. $y' = \dfrac{2x + 3y}{x}$

In Exercises 41–44, find the particular solution that satisfies the initial condition.

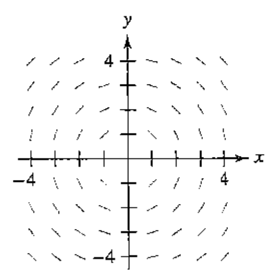
Differential Equation	*Initial Condition*
41. $x\,dy - (2xe^{-y/x} + y)\,dx = 0$	$y(1) = 0$
42. $-y^2\,dx + x(x + y)\,dy = 0$	$y(1) = 1$
43. $\left(x \sec \dfrac{y}{x} + y\right) dx - x\,dy = 0$	$y(1) = 0$
44. $(2x^2 + y^2)\,dx + xy\,dy = 0$	$y(1) = 0$

Slope Fields **In Exercises 45–48, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

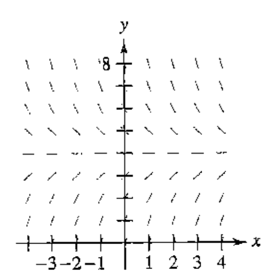
45. $\dfrac{dy}{dx} = x$



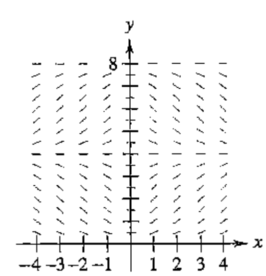
46. $\dfrac{dy}{dx} = -\dfrac{x}{y}$



47. $\dfrac{dy}{dx} = 4 - y$



48. $\dfrac{dy}{dx} = 0.25x(4 - y)$



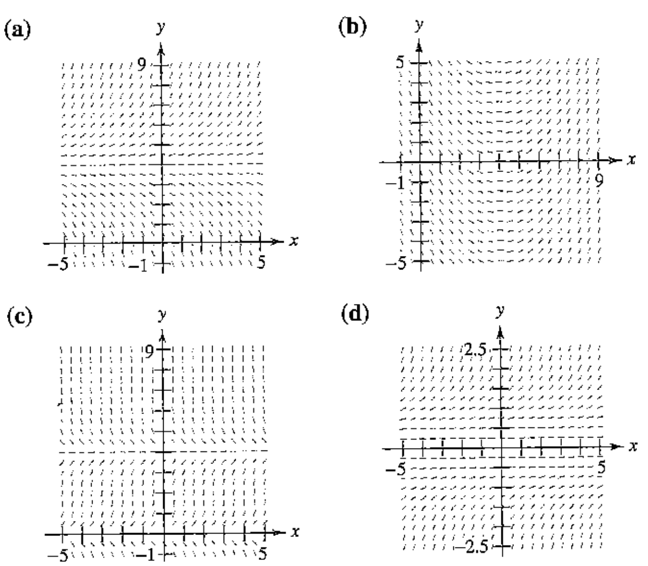
Euler's Method **In Exercises 49–52, (a) use Euler's Method with a step size of $h = 0.1$ to approximate the particular solution of the initial value problem at the given x-value, (b) find the exact solution of the differential equation analytically, and (c) compare the solutions at the given x-value.**

	Differential Equation	*Initial Condition*	*x-value*
49.	$\dfrac{dy}{dx} = -6xy$	$(0, 5)$	$x = 1$
50.	$\dfrac{dy}{dx} + 6xy^2 = 0$	$(0, 3)$	$x = 1$
51.	$\dfrac{dy}{dx} = \dfrac{2x + 12}{3y^2 - 4}$	$(1, 2)$	$x = 2$
52.	$\dfrac{dy}{dx} = 2x(1 + y^2)$	$(1, 0)$	$x = 1.5$

53. ***Radioactive Decay*** The rate of decomposition of radioactive radium is proportional to the amount present at any time. The half-life of radioactive radium is 1599 years. What percent of a present amount will remain after 25 years?

54. ***Chemical Reaction*** In a chemical reaction, a certain compound changes into another compound at a rate proportional to the unchanged amount. If initially there are 20 grams of the original compound, and there is 16 grams after 1 hour, when will 75 percent of the compound be changed?

Slope Fields **In Exercises 55–58, (a) write a differential equation for the statement, (b) match the differential equation with a possible slope field, and (c) verify your result by using a graphing utility to graph a slope field for the differential equation. [The slope fields are labeled (a), (b), (c), and (d).] To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**



55. The rate of change of y with respect to x is proportional to the difference between y and 4.

56. The rate of change of y with respect to x is proportional to the difference between x and 4.

57. The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4.

58. The rate of change of y with respect to x is proportional to y^2.

59. ***Weight Gain*** A calf that weighs 60 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = k(1200 - w)$$

where w is weight in pounds and t is time in years. Solve the differential equation.

(a) Use a computer algebra system to solve the differential equation for $k = 0.8$, 0.9, and 1. Graph the three solutions.

(b) If the animal is sold when its weight reaches 800 pounds, find the time of sale for each of the models in part (a).

(c) What is the maximum weight of the animal for each of the models?

60. ***Weight Gain*** A calf that weighs w_0 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = 1200 - w$$

where w is weight in pounds and t is time in years. Solve the differential equation.

In Exercises 61–66, find the orthogonal trajectories of the family. Use a graphing utility to graph several members of each family.

61. $x^2 + y^2 = C$

62. $x^2 - 2y^2 = C$

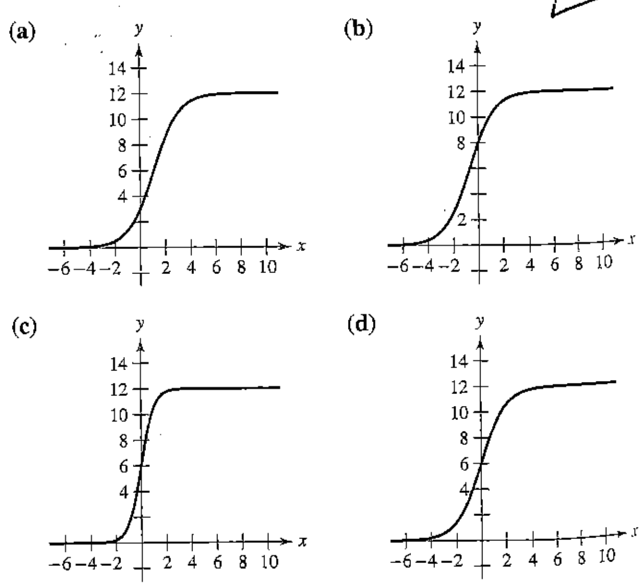
63. $x^2 = Cy$

64. $y^2 = 2Cx$

65. $y^2 = Cx^3$

66. $y = Ce^x$

In Exercises 67–70, match the logistic equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



67. $y = \dfrac{12}{1 + e^{-x}}$

68. $y = \dfrac{12}{1 + 3e^{-x}}$

69. $y = \dfrac{12}{1 + \frac{1}{2}e^{-x}}$

70. $y = \dfrac{12}{1 + e^{-2x}}$

In Exercises 71 and 72, the logistic equation models the growth of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.

71. $P(t) = \dfrac{1500}{1 + 24e^{-0.75t}}$

72. $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

In Exercises 73 and 74, the logistic differential equation models the growth rate of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) use a computer algebra system to graph a slope field, and (d) determine the value of P at which the population growth rate is the greatest.

73. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$

74. $\dfrac{dP}{dt} = 0.1P - 0.0004P^2$

In Exercises 75–78, find the logistic equation that satisfies the initial condition.

	Logistic Differential Equation	*Initial Condition*
75.	$\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{40}\right)$	$(0, 8)$
76.	$\dfrac{dy}{dt} = 1.2y\left(1 - \dfrac{y}{8}\right)$	$(0, 5)$
77.	$\dfrac{dy}{dt} = \dfrac{4y}{5} - \dfrac{y^2}{150}$	$(0, 8)$
78.	$\dfrac{dy}{dt} = \dfrac{3y}{20} - \dfrac{y^2}{1600}$	$(0, 15)$

79. ***Endangered Species*** A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

(a) Write a logistic equation that models the population of panthers in the preserve.

(b) Find the population after 5 years.

(c) When will the population reach 100?

(d) Write a logistic differential equation that models the growth rate of the panther population. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

(e) At what time is the panther population growing most rapidly? Explain.

80. ***Bacteria Growth*** At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 2 grams. The maximum weight of the culture is 10 grams.

(a) Write a logistic equation that models the weight of the bacterial culture.

(b) Find the culture's weight after 5 hours.

(c) When will the culture's weight reach 8 grams?

(d) Write a logistic differential equation that models the growth rate of the culture's weight. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

(e) At what time is the culture's weight increasing most rapidly? Explain.

Writing About Concepts

81. In your own words, describe how to recognize and solve differential equations that can be solved by separation of variables.

82. State the test for determining if a differential equation is homogeneous. Give an example.

83. In your own words, describe the relationship between two families of curves that are mutually orthogonal.

84. ***Sailing*** Ignoring resistance, a sailboat starting from rest accelerates (dv/dt) at a rate proportional to the difference between the velocities of the wind and the boat.

(a) The wind is blowing at 20 knots, and after 1 minute the boat is moving at 5 knots. Write the velocity v as a function of time t.

(b) Use the result of part (a) to write the distance traveled by the boat as a function of time.

True or False? **In Exercises 85–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

85. The function $y = 0$ is always a solution of a differential equation that can be solved by separation of variables.

86. The differential equation $y' = xy - 2y + x - 2$ can be written in separated variables form.

87. The function $f(x, y) = x^2 + xy + 2$ is homogeneous.

88. The families $x^2 + y^2 = 2Cy$ and $x^2 + y^2 = 2Kx$ are mutually orthogonal.

89. Show that if $y = \dfrac{1}{1 + be^{-kt}}$, then $\dfrac{dy}{dt} = ky(1 - y)$.

Putnam Exam Challenge

90. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

This problem was composed by the Committee on the Putnam Prize Competition.

Section 6.4 First-Order Linear Differential Equations

- **Solve a first-order linear differential equation.**
- **Solve a Bernoulli differential equation.**
- **Use linear differential equations to solve applied problems.**

First-Order Linear Differential Equations

In this section, you will see how to solve a very important class of first-order differential equations—first-order linear differential equations.

> **Definition of First-Order Linear Differential Equation**
>
> A first-order linear differential equation is an equation of the form
>
> $$\frac{dy}{dx} + P(x)y = Q(x)$$
>
> where P and Q are continuous functions of x. This first-order linear differential equation is said to be in **standard form.**

NOTE It is instructive to see why the integrating factor helps solve a linear differential equation of the form $y' + P(x)y = Q(x)$. When both sides of the equation are multiplied by the integrating factor $u(x) = e^{\int P(x)\,dx}$, the left-hand side becomes the derivative of a product.

$$y'e^{\int P(x)\,dx} + P(x)ye^{\int P(x)\,dx} = Q(x)e^{\int P(x)\,dx}$$
$$\left[ye^{\int P(x)\,dx}\right]' = Q(x)e^{\int P(x)\,dx}$$

Integrating both sides of this second equation and dividing by $u(x)$ produces the general solution.

To solve a linear differential equation, write it in standard form to identify the functions $P(x)$ and $Q(x)$. Then integrate $P(x)$ and form the expression

$$u(x) = e^{\int P(x)\,dx} \qquad \text{Integrating factor}$$

which is called an **integrating factor.** The general solution of the equation is

$$y = \frac{1}{u(x)}\int Q(x)u(x)\,dx. \qquad \text{General solution}$$

EXAMPLE 1 Solving a Linear Differential Equation

Find the general solution of

$$y' + y = e^x.$$

Solution

For this equation, $P(x) = 1$ and $Q(x) = e^x$. So, the integrating factor is

$$\begin{aligned} u(x) &= e^{\int P(x)\,dx} && \text{Integrating factor} \\ &= e^{\int dx} \\ &= e^x. \end{aligned}$$

This implies that the general solution is

$$\begin{aligned} y &= \frac{1}{u(x)}\int Q(x)u(x)\,dx \\ &= \frac{1}{e^x}\int e^x(e^x)\,dx \\ &= e^{-x}\left(\frac{1}{2}e^{2x} + C\right) \\ &= \frac{1}{2}e^x + Ce^{-x}. && \text{General solution} \end{aligned}$$

THEOREM 6.3 Solution of a First-Order Linear Differential Equation

An integrating factor for the first-order linear differential equation

$$y' + P(x)y = Q(x)$$

is $u(x) = e^{\int P(x)\,dx}$. The solution of the differential equation is

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}\,dx + C.$$

ANNA JOHNSON PELL WHEELER (1883–1966)

Anna Johnson Pell Wheeler was awarded a master's degree from the University of Iowa for her thesis *The Extension of Galois Theory to Linear Differential Equations* in 1904. Influenced by David Hilbert, she worked on integral equations while studying infinite linear spaces.

STUDY TIP Rather than memorizing the formula in Theorem 6.3, just remember that multiplication by the integrating factor $e^{\int P(x)\,dx}$ converts the left side of the differential equation into the derivative of the product $ye^{\int P(x)\,dx}$.

EXAMPLE 2 Solving a First-Order Linear Differential Equation

Find the general solution of

$$xy' - 2y = x^2.$$

Solution The standard form of the given equation is

$$y' + P(x)y = Q(x)$$

$$y' - \left(\frac{2}{x}\right)y = x. \qquad \text{Standard form}$$

So, $P(x) = -2/x$, and you have

$$\int P(x)\,dx = -\int \frac{2}{x}\,dx$$

$$= -\ln x^2$$

$$e^{\int P(x)\,dx} = e^{-\ln x^2}$$

$$= \frac{1}{e^{\ln x^2}}$$

$$= \frac{1}{x^2}. \qquad \text{Integrating factor}$$

So, multiplying each side of the standard form by $1/x^2$ yields

$$\frac{y'}{x^2} - \frac{2y}{x^3} = \frac{1}{x}$$

$$\frac{d}{dx}\left[\frac{y}{x^2}\right] = \frac{1}{x}$$

$$\frac{y}{x^2} = \int \frac{1}{x}\,dx$$

$$\frac{y}{x^2} = \ln|x| + C$$

$$y = x^2(\ln|x| + C). \qquad \text{General solution}$$

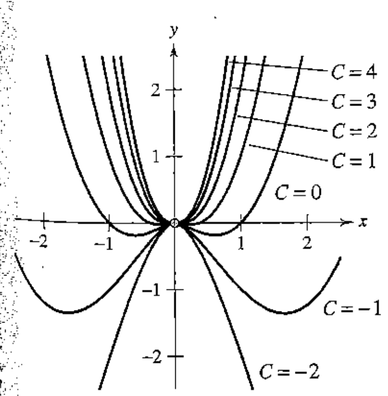


Figure 6.19

Several solution curves (for $C = -2, -1, 0, 1, 2, 3,$ and 4) are shown in Figure 6.19.

EXAMPLE 3 Solving a First-Order Linear Differential Equation

Find the general solution of

$$y' - y \tan t = 1, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Solution The equation is already in the standard form $y' + P(t)y = Q(t)$. S
$P(t) = -\tan t$, and

$$\int P(t)\,dt = -\int \tan t\,dt = \ln|\cos t|$$

which implies that the integrating factor is

$$\begin{aligned} e^{\int P(t)\,dt} &= e^{\ln|\cos t|} \\ &= |\cos t|. \end{aligned}$$ Integrating factor

A quick check shows that $\cos t$ is also an integrating factor. So, multiplyi
$y' - y \tan t = 1$ by $\cos t$ produces

$$\begin{aligned} \frac{d}{dt}[y \cos t] &= \cos t \\ y \cos t &= \int \cos t\,dt \\ y \cos t &= \sin t + C \\ y &= \tan t + C \sec t. \end{aligned}$$ General solution

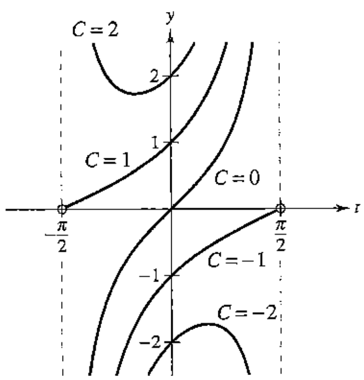


Figure 6.20

Several solution curves are shown in Figure 6.20.

Bernoulli Equation

A well-known nonlinear equation that reduces to a linear one with an appropri
substitution is the **Bernoulli equation,** named after James Bernoulli (1654–1705).

$$y' + P(x)y = Q(x)y^n$$ Bernoulli equation

This equation is linear if $n = 0$, and has separable variables if $n = 1$. So, in t
following development, assume that $n \neq 0$ and $n \neq 1$. Begin by multiplying by y
and $(1 - n)$ to obtain

$$\begin{aligned} y^{-n}y' + P(x)y^{1-n} &= Q(x) \\ (1-n)y^{-n}y' + (1-n)P(x)y^{1-n} &= (1-n)Q(x) \\ \frac{d}{dx}[y^{1-n}] + (1-n)P(x)y^{1-n} &= (1-n)Q(x) \end{aligned}$$

which is a linear equation in the variable y^{1-n}. Letting $z = y^{1-n}$ produces the line
equation

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

Finally, by Theorem 6.3, the *general solution of the Bernoulli equation* is

$$y^{1-n}e^{\int(1-n)P(x)\,dx} = \int (1-n)Q(x)e^{\int(1-n)P(x)\,dx}\,dx + C.$$

EXAMPLE 7 An Electric Circuit Problem

Find the current I as a function of time t (in seconds), given that I satisfies differential equation $L(dI/dt) + RI = \sin 2t$, where R and L are nonzero constants

Solution In standard form, the given linear equation is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}\sin 2t.$$

Let $P(t) = R/L$, so that $e^{\int P(t)\,dt} = e^{(R/L)t}$, and, by Theorem 6.3,

$$Ie^{(R/L)t} = \frac{1}{L}\int e^{(R/L)t}\sin 2t\,dt$$

$$= \frac{1}{4L^2 + R^2}e^{(R/L)t}(R\sin 2t - 2L\cos 2t) + C.$$

So the general solution is

$$I = e^{-(R/L)t}\left[\frac{1}{4L^2 + R^2}e^{(R/L)t}(R\sin 2t - 2L\cos 2t) + C\right]$$

$$I = \frac{1}{4L^2 + R^2}(R\sin 2t - 2L\cos 2t) + Ce^{-(R/L)t}.$$

TECHNOLOGY The integral in Example 7 was found using symbolic algebra software. If you have access to *Derive*, *Maple*, *Mathcad*, *Mathematica*, or the *TI-89*, try using it to integrate

$$\frac{1}{L}\int e^{(R/L)t}\sin 2t\,dt.$$

In Chapter 8 you will learn how to integrate functions of this type using integration by parts.

Exercises for Section 6.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, determine whether the differential equation is linear. Explain your reasoning.

1. $x^3y' + xy = e^x + 1$
2. $2xy - y'\ln x = y$
3. $y' + y\cos x = xy^2$
4. $\dfrac{1 - y'}{y} = 3x$

In Exercises 5–14, solve the first-order linear differential equation.

5. $\dfrac{dy}{dx} + \left(\dfrac{1}{x}\right)y = 3x + 4$
6. $\dfrac{dy}{dx} + \left(\dfrac{2}{x}\right)y = 3x + 2$
7. $y' - y = 10$
8. $y' + 2xy = 4x$
9. $(y + 1)\cos x\,dx - dy = 0$
10. $(y - 1)\sin x\,dx - dy = 0$
11. $(x - 1)y' + y = x^2 - 1$
12. $y' + 3y = e^{3x}$
13. $y' - 3x^2y = e^{x^3}$
14. $y' - y = \cos x$

Slope Fields **In Exercises 15 and 16, (a) sketch an approximate solution of the differential equation satisfying the initial condition by hand on the slope field, (b) find the particular solution that satisfies the initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the hand-drawn graph of part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

	Differential Equation	*Initial Condition*
15.	$\dfrac{dy}{dx} = e^x - y$	$(0, 1)$
16.	$y' + \left(\dfrac{1}{x}\right)y = \sin x^2$	$(\sqrt{\pi}, 0)$

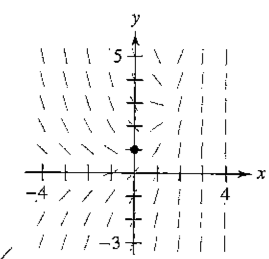


Figure for 15

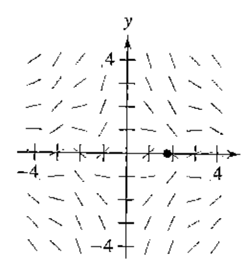


Figure for 16

In Exercises 17–24, find the particular solution of the differential equation that satisfies the boundary condition.

	Differential Equation	*Boundary Condition*
17.	$y'\cos^2 x + y - 1 = 0$	$y(0) = 5$
18.	$x^3y' + 2y = e^{1/x^2}$	$y(1) = e$
19.	$y' + y\tan x = \sec x + \cos x$	$y(0) = 1$
20.	$y' + y\sec x = \sec x$	$y(0) = 4$
21.	$y' + \left(\dfrac{1}{x}\right)y = 0$	$y(2) = 2$
22.	$y' + (2x - 1)y = 0$	$y(1) = 2$
23.	$x\,dy = (x + y + 2)\,dx$	$y(1) = 10$
24.	$2xy' - y = x^3 - x$	$y(4) = 2$